# BIPARTITE $S_2$ GRAPHS ARE COHEN-MACAULAY

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Abstract. In this paper we show that if the Stanley-Reisner ring of the simplicial complex of independent sets of a bipartite graph $G$ satisfies Serre's condition $S_2$, then $G$ is Cohen-Macaulay. As a consequence, the characterization of Cohen-Macaulay bipartite graphs due to Herzog and Hibi carries over this family of bipartite graphs. We check that the equivalence of Cohen-Macaulay property and the condition $S_2$ is also true for chordal graphs and we classify cyclic graphs with respect to the condition $S_2$.

## Introduction

Let $k$ be a field. To any finite simple graph $G$ with vertex set $\mathrm{V} = [n] = \{1, \cdots, n\}$ and edge set $\mathrm{E}(G)$ one associates an ideal $I(G) \subset k[x_1, \cdots, x_n]$ generated by all monomials $x_i x_j$ such that $\{i, j\} \in \mathrm{E}(G)$. The ideal $I(G)$ and the quotient ring $k[x_1, \cdots, x_n]/I(G)$ are called the edge ideal of $G$ and the edge ring of $G$, respectively. The simplicial complex of $G$ is defined by

$$\Delta_G = \{A \subseteq \mathrm{V} | A \text{ is an independent set in } G\},$$

where $A$ is an independent set in $G$ if none of its elements are adjacent. Note that $\Delta_G$ is precisely the simplicial complex with the Stanley-Reisner ideal $I(G)$.

A graph $G$ is said to be Cohen-Macaulay (resp. Buchsbaum) over $k$, if the edge ring of $G$ $k[x_1, \cdots, x_n]/I(G)$ is Cohen-Macaulay (resp. Buchsbaum), and is called Cohen-Macaulay (resp. Buchsbaum) if it is Cohen-Macaulay (resp. Buchsbaum) over any field. A graph is said to be chordal if each cycle of length $> 3$ has a chord.

Let $\Delta$ be a simplicial complex. This complex is called disconnected if the vertex set $V$ of $\Delta$ is the disjoint union of two nonempty sets $V_1$ and $V_2$ such that no face of $\Delta$ has vertices in both $V_1$ and $V_2$, otherwise it is called connected. A simplicial complex $\Delta$ is called Cohen-Macaulay (resp. Buchsbaum) over an infinite field $k$ if its Stanley-Reisner ring $k[\Delta]$ is Cohen-Macaulay (resp. Buchsbaum).

It is known that if $\Delta$ is a disconnected simplicial complex, then $\operatorname{depth} k[\Delta] = 1$, [1, Chapter 5, Ex. 5.1.26]. This implies that if $\operatorname{depth} k[\Delta] > 1$, then $\Delta$ is connected. In particular, every Cohen-Macaulay simplicial complex of positive dimension is connected.

A satisfactory classification of all Cohen-Macaulay graphs over a field $k$ has been standing open for some time. However, as pointed out by Herzog et al [6, Introduction], this is equivalent to a classification of all Cohen-Macaulay simplicial complexes over $k$ which is clearly a hard problem. Accordingly, it is natural to

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study special families of Cohen-Macaulay graphs. Recall that a graph $G$ on the vertex set $[n]$ is bipartite if there exists a partition $[n] = V \cup W$ with $V \cap W = \varnothing$ such that each edge of $G$ is of the form $\{i, j\}$ with $i \in V$ and $j \in W$. It is easy to see that a graph $G$ is bipartite if and only if it has no cycle of odd length. For a Cohen-Macaulay bipartite graph $G$, Estrada and Villarreal [2] showed that $G \setminus \{\nu\}$ is Cohen-Macaulay for some vertex $\nu \in \mathrm{V}(G)$. In [10] it is shown that the cyclic graph $C_n$ is Cohen-Macaulay if and only if $n \in \{3, 5\}$. Herzog and Hibi gave a graph-theoretic characterization of all bipartite Cohen-Macaulay graphs. Due to our direct application, we state their result.

**Theorem** [5, Theorem 3.4]. Let $G$ be a bipartite graph with vertex partition $V \cup W$. Then the following conditions are equivalent:

(a) $G$ is a Cohen-Macaulay graph;
(b) $|V| = |W|$ and the vertices $V = \{x_1, \cdots, x_n\}$ and $W = \{y_1, \cdots, y_n\}$ can be labeled such that:
   (i) $\{x_i, y_i\}$ are edges for $i = 1, \cdots, n$;
   (ii) if $\{x_i, y_j\}$ is an edge, then $i \leq j$;
   (iii) if $\{x_i, y_j\}$ and $\{x_j, y_k\}$ are edges, then $\{x_i, y_k\}$ is also an edge.

Note that this result is characteristic-free.

Let $G$ be a graph with vertex set $\mathrm{V}(G)$ and edge set $\mathrm{E}(G)$. A subset $C \subseteq \mathrm{V}(G)$ is a *minimal vertex cover* of $G$ if: (1) every edge of $G$ is incident with a vertex in $C$, and (2) there is no proper subset of $C$ with the first property. Observe that a minimal vertex cover is the set of indeterminates which generate a minimal prime ideal in the prime decomposition of $I(G)$. Also note that $C$ is a minimal vertex cover if and only if $\mathrm{V}(G) \setminus C$ is a maximal independent set, i.e., a facet of $\Delta_G$.

A graph $G$ is called *unmixed* if all minimal vertex covers of $G$ have the same number of elements, i.e., $\Delta_G$ is pure. It is well known that every Cohen-Macaulay graph $G$ is unmixed. A graph is called chordal if every cycle of length $> 3$ has a chord. Recall that a chord of a cycle is an edge which joins two vertices of the cycle but is not itself an edge of the cycle.

Recall that a finitely generated graded module $M$ over a Noetherian graded $k$-algebra $R$ is said to satisfy the Serre's condition $S_n$ if

$$\operatorname{depth} M_{\mathfrak{p}} \geq \min(n, \dim M_{\mathfrak{p}}),$$

for all $\mathfrak{p} \in \operatorname{Spec}(R)$. Thus, $M$ is Cohen-Macaulay if and only if it satisfies the Serre's condition $S_n$ for all $n$. A graph is said to satisfy the Serre's condition $S_n$, or simply is an $S_n$ graph, if its edge ring satisfies this condition. Using [7, Lemma 3.2.1] and Hochster's formula on local cohomology modules, a pure $d$-dimensional Stanley-Reisner ring $k[\Delta]$ satisfies $S_2$ property if and only if $\widetilde{H}_0(\mathrm{link}_\Delta(F); k) = 0$ for all $F \in \Delta$ with $|F| \leq d - 2$ (see [8, page 4]).

The main result of this paper is to prove that if $G$ is a bipartite $S_2$ graph, then $G$ is Cohen-Macaulay (see Theorem 1.3). Consequently, the characterization of Cohen-Macaulay bipartite graphs by Herzog and Hibi carries over bipartite $S_2$ graphs. It is shown that not only for bipartite graphs but also for chordal graphs Cohen-Macaulay property and the condition $S_2$ are equivalent. To see an example of a non-Cohen-Macaulay $S_2$ graph, it is shown that the cyclic graph $C_n$ of length $n \geq 3$ is $S_2$ if and only if $n = 3, 5$ or $7$. In particular, $C_7$ is the only cyclic graph which is $S_2$ but not Cohen-Macaulay. Finally, we reprove some known results

on certain bipartite Cohen-Macaulay graphs by providing rather simpler proofs compared to the existing ones.

## 1. The Main Result

Our results are inspired by the aforementioned theorem of Herzog and Hibi [5, Theorem 3.4].

**Proposition 1.1.** *Let $G$ be an unmixed bipartite graph with bipartition $V = \{x_1, \cdots, x_n\}$ and $W = \{y_1, \cdots, y_n\}$ such that $\{x_i, y_i\}$ is an edge of $G$ for all $i = 1, \cdots, n$. Then $V$ and $W$ can be simultaneously relabeled such that the following statements are equivalent:*

(a) *There exists a linear order $V = F_0, ..., F_n = W$ on some of the facets of $\Delta_G$ such that $F_i$ and $F_{i+1}$ intersect in codimension one for $i = 0, \cdots, n-1$.*
(b) *If $\{x_i, y_j\}$ is an edge, then $i \leq j$.*

By a simultaneous relabeling we mean that for all $i$, $x_i$ and $y_i$ receive the same relabeling. In particular, under the assumptions of Proposition 1.1, with the new labeling, $\{x_i, y_i\}$ is an edge of $G$ for all $i = 1, \cdots, n$.

Before proceeding on the proof of this Proposition note that the condition (a) is weaker than strongly connectedness of $\Delta_G$. Recall that a simplicial complex $\Delta$ is strongly connected if for any two facets $V$ and $W$ of $\Delta$ there exists a chain of facets satisfying (a). Here we only need this sequence just for the two specific facets $V$ and $W$.

*Proof.* (a)$\Rightarrow$(b): We have $|F_1 \backslash F_0| = 1$, say $F_1 \backslash F_0 = \{y_1\}$. Then $F_1 = \{y_1, x_2, \cdots, x_n\}$ because $\{x_1, y_1\}$ is not a face of $\Delta_G$. Similarly, $|F_2 \setminus F_1| = 1$, say $F_2 \setminus F_1 = \{y_2\}$. Thus $F_2 = \{y_1, y_2, x_3, \cdots, x_n\}$ because again $\{x_2, y_2\}$ is not a face of $\Delta_G$. Hence by induction we may assume that $F_i = \{y_1, \cdots, y_i, x_{i+1}, \cdots, x_n\}$ for $i = 0, \cdots, n$. In particular, if $i > j$, then $\{x_i, y_j\}$ is a face of $\Delta_G$, and hence it is not an edge of $G$.

(b)$\Rightarrow$(a): Set $F_i = \{y_1, \cdots, y_i, x_{i+1}, \cdots, x_n\}$. It is easy to see that for any $i$, $F_i$ is a maximal independent set and hence a facet of $\Delta_G$. Moreover $F_i$ and $F_{i+1}$ intersect in codimension one.

□

**Lemma 1.2.** *Let $G$ be a bipartite graph. Then $G$ is a non-complete bipartite graph if and only if $\Delta_G$ is connected.*

*Proof.* Let $V_1 \cup V_2$ be the bipartition of $G$. Then $G$ fails to be a complete bipartite graph if and only if there are two vertices $x \in V_1$ and $y \in V_2$ which are not adjacent, that is, $\{x, y\}$ is an independent set of $G$, i.e., $\Delta_G$ is connected. □

Now we may state the main result which in particular provides a characterization of bipartite $S_2$ graphs.

**Theorem 1.3.** *Let $G$ be a bipartite graph with at least four vertices and with vertex partition $V$ and $W$. Then the following are equivalent:*

(a) *$G$ is unmixed and $V$ and $W$ can be labeled such that there exists an order $V = F_0, \cdots, F_n = W$ of the facets of $\Delta_G$ where $F_i$ and $F_{i+1}$ intersect in codimension one for $i = 0, \cdots, n-1$.*
(b) *$G$ is a Cohen-Macaulay graph.*

(c) *$G$ is a Buchsbaum non-complete bipartite graph.*
(d) *$G$ is an $S_2$ graph.*

*Proof.* We prove $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (a)$.

(a)⇒(b): Since $G$ is unmixed, by König's Theorem there is a bipartition $V = \{x_1, \cdots, x_n\}$ and $W = \{y_1, \cdots, y_n\}$ such that $\{x_i, y_i\}$ is an edge of $G$ for all $i$. By Proposition 1.1, $V$ and $W$ can be relabeled such that $\{x_i, y_i\}$ is an edge of $G$ for all $i$ and if $\{x_i, y_j\}$ is an edge in $G$, then $i \leq j$. We fix such a labeling. Let $\{x_i, y_j\}$ and $\{x_j, y_k\}$ be edges of $G$ with $i < j < k$, and suppose that $\{x_i, y_k\}$ is not an edge of $G$. Since $\{x_i, y_k\}$ is a face of $\Delta_G$ and $G$ is unmixed, $\Delta_G$ is pure, hence there exists a facet $F$ of $\Delta_G$ with $|F| = n$ and $\{x_i, y_k\} \subset F$. Since $F$ is a facet of $\Delta_G$, any 2-element subset of $F$ is a non-edge of $G$. We have $y_j \notin F$ since $\{x_i, y_j\}$ is an edge of $G$. Similarly $x_j \notin F$ since $\{x_j, y_k\}$ is an edge of $G$. On the other hand, since $\{x_t, y_t\}$ is an edge of $G$ for all $t$, the facet $F$ can not contain both $x_t$ and $y_t$. Hence $F$ is of the form $F = \{z_1, \cdots, z_n\}$, where $z_t = x_t$ or $y_t$ for $t = 1, \cdots, n$. Thus either $y_j$ or $x_j$ belongs to $F$, which is a contradiction. Consequently, $G$ is Cohen-Macaulay by the theorem of Herzog and Hibi.

(b)⇒(c): Since every Cohen-Macaulay ring is a Buchsbaum ring, $G$ is also Buchsbaum. By definition, the ideal of the simplicial complex $\Delta_G$ is equal to edge ideal of $G$. Hence $\Delta_G$ is also Cohen-Macaulay and in particular, $\Delta_G$ is connected. Therefore, by Lemma 1.2 $G$ is non-complete.

(c)⇒(d): By [11, Corollary 2.7] the localization of every Buchsbaum ring at any of its prime ideals which is not equal to $(x_1, \cdots, x_n, y_1, \cdots, y_n)$, is Cohen-Macaulay. Therefore $G$ satisfies the $S_2$ condition.

(d)⇒(a): Since $\Delta_G$ satisfies the $S_2$ condition, by [4, Corollary 2.4] for any two facets $F$ and $H$ of $\Delta_G$, there exist a positive integer $m$ and a sequence $F = F_0, \cdots, F_m = H$ of facets of $\Delta_G$ such that $F_i$ intersects $F_{i+1}$ in codimension one for all $i = 0, \cdots, m-1$. Hence $\Delta_G$ is strongly connected. In particular, since the partitions $V$ and $W$ of the vertices of $G$ can be considered as two facets of $\Delta_G$ and $\Delta_G$ is strongly connected, the required sequence exists. Furthermore, $\mid F_i \mid = \mid F_i \cap F_{i+1} \mid +1 = \mid F_{i+1} \mid$ for all $i = 0, \cdots, m-1$. This implies that any two facets of $\Delta_G$ have the same number of elements and hence $G$ is unmixed. □

**Remark 1.4.** The implication (b)⇒(a) in the above theorem does not depend on the bipartite assumption of $G$ and is valid in a more general setting. In fact a stronger implication is valid. More precisely, every Cohen-Macaulay simplicial complex is strongly connected. This follows, for example, by an argument similar to the implication (d)⇒(a).

**Remark 1.5.** Theorem 1.3 reveals that for bipartite graphs Cohen-Macaulay and $S_2$ properties are equivalent. This raises the question whether there are other families of graphs for which these two properties are equivalent. Here, we show that,

(1) Every chordal $S_2$ graph is Cohen-Macaulay.
(2) The cyclic graph $C_7$ is $S_2$ but not Cohen-Macaulay.

In fact, chordal graphs are shellable [9, Theorem 2.13]. But any $S_2$ graph is unmixed (see [3, Corollary 5.10.9], or [4, Remark 2.4.1]). Therefore, for chordal graphs Cohen-Macaulay and $S_2$ properties are equivalent.

To establish (2) we classify all cyclic graphs $C_n$ with respect to $S_2$ property.

**Proposition 1.6.** *The cyclic graph $C_n$ of length $n \geq 3$ is $S_2$ if and only if $n = 3, 5$ or $7$. In particular, $C_7$ is the only cyclic graph which is $S_2$ but not Cohen-Macaulay.*

*Proof.* It is known that $C_n$ is Cohen-Macaulay if and only if $n = 3, 5$ [10, Corollary 6.3.6]. On the other hand, $C_n$ of length $n \geq 3$ is unmixed if and only if $n = 3, 4, 5, 7$ [10, Exercise 6.2.15]. Accordingly, $C_3$ and $C_5$ are $S_2$. Since $C_4$ is bipartite but not Cohen-Macaulay, by Theorem 1.3 it is not $S_2$. Furthermore, as mentioned before, every $S_2$ graph is unmixed. Thus, the only cyclic graph which remains to be checked is $G = C_7$. To settle this, we apply the cohomological criterion for $S_2$ property mentioned in the introduction. In fact, we need to check that for all $F \in \Delta_G$ with $|F| \leq 1$, $\widetilde{H}_0(\mathrm{link}_{\Delta_G}(F); k) = 0$. This condition is satisfied if $\mathrm{link}_{\Delta_G}(F)$ is connected which can easily be checked by direct inspection. □

In light of Theorem 1.3, we consider some known results on certain bipartite Cohen-Macaulay graphs and we provide rather simpler proofs compared to the existing ones.

As a consequence of Theorem 1.3(b) we may state the following result on the structure of trees satisfying the condition $S_2$.

**Corollary 1.7.** [10, Theorem 6.3.4] *Let $G$ be a tree with at least four vertices. Then the following are equivalent:*

(a) *$G$ satisfies the condition $S_2$.*
(b) *There is a bipartition $V = \{x_1, \cdots, x_n\}$, $W = \{y_1, \cdots, y_n\}$ of $G$ such that*
  *(i) $\{x_i, y_i\} \in E(G)$ for all $i$.*
  *(ii) for each $i$ either $\deg(x_i) = 1$ or $\deg(y_i) = 1$, exclusively.*
  *(iii) $V$ and $W$ can be simultaneously relabeled such that there exists an order $V = F_0, \cdots, F_n = W$ of the facets of $\Delta_G$ where $F_i$ and $F_{i+1}$ intersect in codimension one for $i = 0, \cdots, n-1$.*

From part (b)(*ii*) of Corollary 1.7 it follows that every tree with $2n$ vertices which satisfies the condition $S_2$, has precisely $n$ vertices of degree one.

**Corollary 1.8.** *Every path of length greater than four does not satisfy the condition $S_2$ and hence it is not Cohen-Macaulay.*

By Corollary 1.7 every bipartite $S_2$ graph has at least two vertices of degree one. From this fact and Theorem 1.3 we get the following result which is a special case of [10, Proposition 6.2.1].

**Proposition 1.9.** *Let $G$ be a bipartite $S_2$ graph. Let $y$ be a vertex of degree one of $G$ and $x$ its adjacent vertex. Then $G \setminus \{x, y\}$ is still an $S_2$ graph.*

*Proof.* Since $G$ is bipartite, there exists an order $V = F_0, \cdots, F_n = W$ of facets of $\Delta_G$ such that for each $i = 0, \cdots, n-1, F_i$ intersects $F_{i+1}$ in codimension one. Since for each $i$, $V \cup W \setminus F_i$ is a minimal vertex cover of $G$, it contains exactly one of the vertices $x$ or $y$. Thus $F_i$ contains $y$ or $x$ respectively. Again since any facet of $\Delta_G$ is an independent set, none of these facets can contain both of these elements. Thus, if we delete both of these elements from $V(G)$, then they will be deleted from each element of the sequence $V = F_0, \cdots, F_n = W$. By construction $F_0 \setminus \{x\} = F_1 \setminus \{y\}$, and hence we obtain a sequence of length $n-1$ of facets of $\Delta_{G\setminus\{x,y\}}$ such that each two consecutive members of this sequence intersect each other in codimension one. Now the claim follows from Theorem 1.3(b).

□

**Remark 1.10.** A careful inspection of the proof of Proposition 1.9 reveals that every edge $\{x, y\}$ where $y$ is an arbitrary degree one vertex of $G$, intersects every member of the sequence $F_0, \cdots, F_n$. Conversely, if we add a new vertex $x_{n+1}$ to $V$ and a new vertex $y_{n+1}$ to $W$ and the edge $\{x_{n+1}, y_{n+1}\}$ to $G$, then the bipartite graph $G_1 = V_1 \cup W_1$, where $V_1 = V \cup \{x_{n+1}\}$ and $W_1 = W \cup \{y_{n+1}\}$, has the sequence $F_0 \cup \{x_{n+1}\}, F_1 \cup \{x_{n+1}\}, \cdots, F_n \cup \{x_{n+1}\}, F_{n+1} = F_n \cup \{y_{n+1}\}$ as a subsequence of its facets which satisfies the assumption of Theorem 1.3(b), hence $G_1$ is an $S_2$ graph.

We end this paper with the following immediate result which is again a special case of [10, Proposition 6.2.1].

**Corollary 1.11.** *Let $G$ be a tree with more than two vertices which is $S_2$. Let $x$ be a degree one vertex of $G$ and $y$ its adjacent vertex. Then $G \setminus \{x, y\}$ is an $S_2$ graph.*

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